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On the exact solution of (2+1)-dimensional cubic nonlinear Schrödinger (NLS) equation

E A Saied, Reda G Abd El-Rahman and Marwa I Ghonamy

Mathematics Department, Faculty of Science, Benha University, Benha, Egypt

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Abstract
In this paper, symmetry reductions for a cubic nonlinear Schrödinger (NLS) equation to complex ordinary differential equations are presented. These are obtained by means of Lie's method of infinitesimal transformation groups. It is shown that ten types of subgroups of the symmetry group lead, via symmetry reduction, to ordinary differential equations. These equations are solved and the similarity solutions are obtained.

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1. Introduction

The two-dimensional cubic nonlinear Schrödinger (NLS) equation may be written as

$$\mathrm{i}\psi_t + c(\psi_{xx} + \psi_{yy}) + a|\psi|^2\psi = 0 \tag{1}$$

where a, c are constants and subscripts x, y, t represent partial derivatives. This type of nonlinear partial differential equation (PDE) occurs in a wide variety of physical applications, and the complex function $\psi(x, y, t)$ has different physical meanings in different branches of physics. For instance, it can be obtained in the contexts of nonlinear optics [1–4], modelling, the propagation of an intense laser beam through a medium with Kerr nonlinearity. In this model $\psi(x, y, t)$ is the electric field amplitude, t is the distance in the direction of propagation, and x and y are the transverse spatial coordinates. It is known that there exist solutions of equation (1) which have a singularity in finite time and are extremely sensitive to the addition of small perturbations to the equation and there has been much interest in the determination of the structure of this singularity [2, 5]. Due to the extremely high optical intensity of a focused ultrashort laser pulse, it will interact with the beam delivery medium through nonlinear optical mechanisms, by which the refractive index of the medium is changed: the Kerr effect results in an increase in the refractive index, while the plasma effect results in a decrease in refractive index from the generation of free electrons through the ionization process [6–9]. In physical self-focusing, an electromagnetic wave is absorbed by the medium through which it propagates, an effect which is neglected in equation (1) which models propagation under 'ideal transparency'. When damping (absorption) is included [10, 11], the model equation becomes

$$\mathrm{i}\psi_t + c(\psi_{xx} + \psi_{yy}) + a|\psi|^2\psi + b\psi = 0 \tag{2}$$

where the constant b plays the role of absorption coefficient, acts as a defocusing mechanism, and depends on some physical parameters (for more details of the physical value of b, see [11]). The complex function $\psi(x, y, t)$ in the mathematical model of NLS equation (2) arises in many physical applications and also $\psi(x, y, t)$ has different physical meanings in different branches of physics. It may be an electromagnetic potential and the NLS equation then describes, for instance, the collapse of Langmuir waves with collisional damping [12]. In other applications, $\psi(x, y, t)$ can be a complex order parameter, describing various physical phenomena close to critical stability, in the context of the complex Ginzburg–Landau equation where b plays the role of the instability parameter [13]. In addition to critical phenomena, a multiplicative Gaussian white noise term is included in the NLS equation (2), and its effect on the coherence of the ground state solitary solution to the unperturbed NLS equation was discussed [14]. The mathematical model of the NLS equation (2) also has numerous applications in the modelling of the dynamics of spatial solitary waves in saturated amplifying/absorbing media [15] and the dynamics of pulse propagation in nonlinear rare-earth-doped optical fibres for which material dispersion, gained dispersion and nonlinearity contribute significantly [16–21]. Because of its wide range of applications, the properties of the NLS equation (2) are a continuous subject of study in both physical and mathematical contexts [22–26]. Among the properties already known, we mention the following: numerical integration of the NLS equation (2) was also performed and led to the determination of coherent structures with complex field profiles [27]. The Hirota method [28] has been used to rewrite the (l + l)-dimensional NLS equation (2) in a bilinear form in order to obtain exact solutions describing solitary waves and shock fronts [29]. A stability criterion which determines whether the system underlying the NLS equation (2) evolves into a monochromatic state was discussed [30]. The NLS equation (2) does not belong to the class of integrable nonlinear evaluation equation [31, 32] even in (l + l) dimensions, still less in (2+1). Thus, no Lax pair exists and no linear techniques are available for solving this equation. Exact solitons and multisolitons are hence not to be expected. The motivation for the present study lies in the physical importance of the model NLS equation (2) and the need to have some exact solutions. To have an explicit analytic solution of equation (2) may enable one to better understand the physical phenomena which it describes. The exact solutions, which are accurate and explicit, may help physicists and engineers to discuss and examine the sensitivity of the model to several important physical parameters. To our knowledge, a detailed analysis that leads to an exact analytic solution for equation (2) has not been performed, and is therefore desirable. There is an abundance of transformations of various types that appear in the literature of mathematics that are generally aimed at obtaining some sort of simplification of partial differential models. Lie's method of infinitesimal transformation groups which essentially reduce the number of independent variables in partial differential equations (PDE) has been widely used in equations of mathematical physics [33–35]. Currently, there is much interest in the determination of symmetry reduction of PDEs, which reduce the original equations to ordinary differential equations (ODEs). One then checks if the resulting ODEs can be solved explicitly, leading to exact solutions of the original PDE. This method represents one of the few systematic methods of obtaining nonlinear PDEs, with an eye to obtaining exact solutions. The basic concepts and equations of the Lie group method were developed and described in various books [36]. The symmetry reduction of some types of NLS equations has been discussed [37–40] by using the Lie group method. Our aim is to obtain some exact solutions of equation (2), in the course of which we will be utilizing the Lie group analysis which exploits the symmetries of equation (2) to derive some ansatz leading to the reduction of variables, where the analytic solutions are easier to obtain and elegant closed form solutions exist, so we confine ourselves to a short introduction coupled with a summary of the main equations. This method consists of several steps.

(i) Find the Lie group of point transformations

$$\begin{aligned} &\bar{x} = x + \varepsilon X(x, y, t, \psi) + O(\varepsilon^2) \qquad && \bar{y} = y + \varepsilon Y(x, y, t, \psi) + O(\varepsilon^2) \\ &\bar{t} = t + \varepsilon T(x, y, t, \psi) + O(\varepsilon^2) && \bar{\psi} = \psi + \varepsilon \Psi(x, y, t, \psi) + O(\varepsilon^2) \end{aligned} \tag{3a}$$

and

$$\begin{aligned} &\bar{\psi}_{\bar{t}} = \psi_t + \varepsilon \Psi^t + O(\varepsilon^2) \qquad \bar{\psi}_{\bar{x}} = \psi_x + \varepsilon \Psi^x + O(\varepsilon^2) \qquad \bar{\psi}_{\bar{y}} = \psi_y + \varepsilon \Psi^y + O(\varepsilon^2) \\ &\bar{\psi}_{\overline{xx}} = \psi_{xx} + \varepsilon \Psi^{xx} + O(\varepsilon^2) \qquad \bar{\psi}_{\overline{yy}} = \psi_{yy} + \varepsilon \Psi^{yy} + O(\varepsilon^2) \end{aligned} \tag{3b}$$

leaving equation (2) invariant. In other words, the transformations (3*a*) are such that $\bar{\psi}(\bar{x}, \bar{y}, \bar{t})$ is a solution, whenever $\psi(x, y, t)$ is one, where the functions Ψ^t, Ψ^x, Ψ^y, Ψ^{xx} and Ψ^{yy} in equation (3*b*) can be determined from equation (3*a*).

(ii) Assuming that equation (2) is invariant under the transformations (3*a*) and (3*b*), we get the following relation from the coefficient of the first order of ε:

$$\mathrm{i}\Psi^t + c(\Psi^{xx} + \Psi^{yy}) + a(2|\psi|^2\Psi + \psi^2\Psi^*) + b\Psi = 0 \tag{4}$$

where the asterisk designates the complex conjugate.

(iii) The general solution of equation (4) gives the infinitesimal elements X, Y, T and Ψ as a function of (x, y, t, ψ) and arbitrary constants.

(iv) Thus, the similarity variables and form can be obtained by solving the characteristic equation

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}\psi}{\Psi}. \tag{5}$$

The general solution of equation (5) involves three constants, two of which (s and r) become new independent variables and the third constant, f, plays the role of a new dependent value. Expressing the dependent variable, ψ, in terms of these constants provides an expression of the type

$$\psi(x, y, t) = g(x, y, t) f(s, r) \tag{6}$$

where g, s and r are known functions of the independent variables x, y and t. It should be noted that similarity variables s, r and form f, obtained from integration of equation (5) are quite different to each other depending on the choice of values of constants in X, Y, T and Ψ.

(v) Substitute (6) into the original equation (2) and obtain the PDE in s, r variables for the function f. Since we have only two variables (s, r), a first step of the symmetry reduction is achieved. We obtain ten types of reduced PDEs which are tabulated in table 1.

(vi) To get reductions of equation (2) to ODEs, we apply once more the procedure mentioned above to PDEs in table 1.

(vii) Solve the reduced ODEs and substitute into equation (2) to obtain exact solutions of the original equation (2). The obtained solutions will be invariant under the considered subgroup of Lie group. While there is no guarantee that we will be able to solve these ODEs analytically for all reductions, in many cases we can find some particular solutions.

Table 1. Similarity variables and similarity functions of equation (2) and the corresponding reduced PDEs of two variables.

Essential vector fields	$s(x,y,t), r(x,y,t), \psi(x,y,t)$	Reduction equations	Type
V_2	$\frac{x}{t}, \frac{y}{t}, \frac{f}{t}\exp\left(\frac{\mathrm{i}}{4ct}(x^2+y^2)+\mathrm{i}bt\right)$	$cf_{rr}+cf_{ss}+a\|f\|^2f=0$	Type 1
V_1	$x^2+y^2, t, f(s,r)$	$\mathrm{i}f_r+4cf_s+4csf_{ss}+ a\|f\|^2f+bf=0$	Type 2
V_5	$x, t, f\exp\left(\frac{\mathrm{i}y^2}{4ct}\right)$	$\mathrm{i}f_r+\frac{\mathrm{i}}{2r}f+cf_{ss}+ a\|f\|^2f+bf=0$	Type 3
$V_6+V_8+V_9$	$x-y, t, f\exp\left(\frac{-\mathrm{i}y}{2c}\right)$	$4cf_{ss}+2\mathrm{i}f_s+2\mathrm{i}f_r+2a\|f\|^2f+ \left(2b-\frac{1}{2c}\right)f=0$	Type 4
dV_8+V_6	$x-dy, t, f$	$c(1+d^2)f_{ss}+\mathrm{i}f_r+ a\|f\|^2f+bf=0$	Type 5
V_2+kV_9	$\frac{x}{t}, \frac{y}{t}, \frac{f}{t}\exp\left(\mathrm{i}\left(\frac{(x^2+y^2)}{4ct}+bt+\frac{k}{ct}\right)\right)$	$cf_{rr}+cf_{ss}+\frac{k}{c}f+ a\|f\|^2f=0$	Type 6
$\lambda V_6+V_4+\sigma V_8$	$x-\sigma t, y-\lambda t, f\exp(\mathrm{i}bt)$	$cf_{ss}-\mathrm{i}\sigma f_s-\mathrm{i}\lambda f_r+cf_{rr}+ a\|f\|^2f=0$	Type 7
V_8+eV_5	$x-\frac{y}{et}, t, f\exp\left(\frac{\mathrm{i}y^2}{4ct}\right)$	$c\left(1+\frac{1}{e^2r^2}\right)f_{ss}+\mathrm{i}f_r+\frac{\mathrm{i}}{2r}f+ a\|f\|^2f+bf=0$	Type 8
$V_8+V_4+V_5$	$x-t, y-\frac{t^2}{2}, f\exp\left(\mathrm{i}\left(bt+\frac{yt}{2c}-\frac{t^3}{6c}\right)\right)$	$2cf_{rr}+2cf_{ss}-2\mathrm{i}f_s-\frac{1}{c}rf+ 2a\|f\|^2f=0$	Type 9
$V_3+V_5+V_7$	$\frac{x}{\sqrt{t}}-2\sqrt{t}, \frac{y}{\sqrt{t}}-2\sqrt{t}, \frac{f}{\sqrt{t}}\exp\left(\mathrm{i}\left(bt+\frac{1}{c}(x+y-2t)\right)\right)$	$2cf_{rr}+2cf_{ss}-\mathrm{i}rf_r-\mathrm{i}f- \mathrm{i}sf_s+2a\|f\|^2f=0$	Type 10

The plan of the paper is as follows. Section 2 is entirely devoted to showing how the powerful Lie group method can be used to generate ten of the symmetry reductions. The reductions are constructed by imposing these symmetries to get ten PDEs of two variables only. In section 3 we present the reduced ordinary differential equations and their exact solutions. Solutions of these ODEs lead by back substitution to a large variety of solutions of NLS equation (2) in explicit form. Section 4 contains discussion and concluding remarks.

2. Symmetry group

In order to find the symmetry group of equation (1), we look for an algebra of vector fields of the form

$$V = T\frac{\partial}{\partial t}+X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+\Psi\frac{\partial}{\partial\psi}+\Psi^x\frac{\partial}{\partial\psi_x}+\Psi^y\frac{\partial}{\partial\psi_y}+\Psi^t\frac{\partial}{\partial\psi_t}+\Psi^{xx}\frac{\partial}{\partial\psi_{xx}}+\Psi^{yy}\frac{\partial}{\partial\psi_{yy}} \tag{7}$$

where all the coefficients in equation (7) are functions of x, y, t and ψ. The coefficients X, Y, T and Ψ are determined from equation (4), by setting the coefficients of different differentials of ψ equal to zero. We obtain a large number of PDEs in X, Y, T and Ψ that need to be satisfied. Therefore, these equations enable us to derive the functions X, Y, T and Ψ and consequently the desired basis for their Lie algebra. Without presenting any calculations, the results can be summarized as follows: the nine linear independent infinitesimal generators which determine the symmetries under which equation (2) is invariant can be spanned by the infinitesimal generators

$$
\begin{aligned}
V_1 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \\
V_2 &= \frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}yt\frac{\partial}{\partial y} + \frac{t^2}{2}\frac{\partial}{\partial t} + \left[\mathrm{i}\left(\frac{x^2}{8c} + \frac{y^2}{8c} + \frac{bt^2}{2}\right) - \frac{t}{2}\right]\psi\frac{\partial}{\partial \psi} \\
V_3 &= \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + \left(\mathrm{i}bt - \frac{1}{2}\right)\psi\frac{\partial}{\partial \psi} \\
V_4 &= \frac{\partial}{\partial t} + \mathrm{i}b\psi\frac{\partial}{\partial \psi} \qquad V_5 = t\frac{\partial}{\partial y} + \frac{\mathrm{i}}{2c}y\psi\frac{\partial}{\partial \psi} \qquad V_6 = \frac{\partial}{\partial y} \\
V_7 &= t\frac{\partial}{\partial x} + \frac{\mathrm{i}}{2c}x\psi\frac{\partial}{\partial \psi} \qquad V_8 = \frac{\partial}{\partial x} \qquad V_9 = \frac{-\mathrm{i}}{2c}\psi\frac{\partial}{\partial \psi}.
\end{aligned}
\tag{8}
$$

As mentioned before, the main use of these generators is to obtain a reduction of variables in equation (2), which can be obtained by solving the characteristic equation (5). Reductions of equation (2) may be obtained from any linear combination

$$a_1V_1 + a_2V_2 + \cdots + a_9V_9 \qquad a_i \in R. \tag{9}$$

Since there is almost an infinite number of such combinations, it is usually not feasible to list all possible similarity reductions.

A systematic procedure of classifying these reductions is based on the property that the transformations of the symmetry group will transform solutions of equation (2) into other solutions. Therefore, it is sufficient to consider only linear combinations, which lead to reductions that are inequivalent with respect to symmetry transformations; this set of solutions is called an optimal system. Precisely, by introducing the adjoint representation of the Lie algebra, we obtain the following basic fields of an optimal system, from which every other solution can be derived,

$$
\begin{aligned}
&V_2,\ V_1,\ V_5,\ V_6+V_8+V_9,\ dV_8+V_6,\ V_2+kV_9,\ \lambda V_6+V_4+\sigma V_8, \\
&\qquad V_8+eV_5,\ V_8+V_4+V_5,\ V_3+V_5+V_7.
\end{aligned}
$$

This produces the essential types of the reduced (2+1)-dimensional cubic NLS equation (2), which are PDEs of the similarity variables s and r, as well as similarity solutions $f(s,r)$; one then finds the ten types of reduced equations listed in table 1.

3. Reductions to ordinary differential equation

In the following, we look for transformations that will reduce the PDEs in table 1 into some ODEs. In this section we will once more apply the procedure of symmetry reduction method to obtain the vector fields of each PDE in table 1, which leads to essential types of ODEs.

The similarity variable and similarity form can be obtained by solving the characteristic equation of the vector fields. The general form involves two constants, one of them (p) becomes a new independent variable and the second constant h, plays the role of a new dependent variable. Expressing the dependent variable f in terms of p and h provides an expression of the type

$$f(s,r) = \beta(s,r)h(p) \tag{10}$$

where β and p are known functions of s and r. Substituting (10) into the PDEs in table 1, one obtains the reduced ODEs. Some exact solutions of each ODE are studied. Now we shall try this method for each type in table 1.

Case 1. Let us consider the PDE of type 1 in table 1,

$$cf_{rr} + cf_{ss} + a|f|^2 f = 0. \tag{11}$$

By applying the procedure mentioned in section 2 to this equation, we have the following infinitesimal generators:

$$B_1 = -s\partial_s - r\partial_r + f\partial_f \qquad B_2 = s\partial_r - r\partial_s$$
$$B_3 = \partial_r \qquad B_4 = \partial_s \qquad B_5 = \mathrm{i} f\partial_f.$$

The essential reductions of PDE (11) to ODEs can be derived by the optimal system

$$B_2, B_1, B_3 + B_4, B_3 + B_5 \text{ and } B_3 + B_4 + B_5.$$

Now, let us consider the vector field $B_2 = s\partial_r - r\partial_s$.

By solving its characteristic equation, the similarity variable and form are given by

$$p = s^2 + r^2 \qquad \text{and} \qquad f = h(p). \tag{12}$$

Substituting equation (12) into equation (11), we have

$$4cph'' + 4ch' + a|h|^2 h = 0. \tag{13}$$

To find a special solution of equation (13), we put

$$h(p) = M(P)\exp(\mathrm{i}G(p)) \tag{14}$$

in equation (13) where *M* and *G* are real functions. We get two coupled real equations

$$2pM'' - 2pMG'^2 + 2M' + \frac{a}{2c}M^3 = 0 \tag{15a}$$

$$2pM'G' + pMG'' + MG' = 0. \tag{15b}$$

For $\frac{c}{a} < 0$, the system (15*a*), (15*b*) has the special solution

$$G = \alpha \qquad M = \sqrt{\frac{-c}{ap}} \tag{16}$$

where α is an arbitrary constant. From (16) and (14), we have

$$h(p) = \sqrt{\frac{-c}{ap}}\exp(\mathrm{i}\alpha) \tag{17a}$$

From equations (17*a*) and (12), we have

$$f(s, r) = \sqrt{\frac{-c}{a(s^2 + r^2)}}\exp(\mathrm{i}\alpha). \tag{17b}$$

From equation (17*b*) and ansatz of type 1, in table 1 where $s = \frac{x}{t}$, $r = \frac{y}{t}$ and $\psi(x, y, t) = \frac{f}{t}\exp\left(\frac{\mathrm{i}}{4ct}(x^2 + y^2) + \mathrm{i}bt\right)$, we get the exact solution

$$\psi(x, y, t) = \sqrt{\frac{-c}{a(x^2 + y^2)}}\exp\left(\mathrm{i}\left(bt + \frac{(x^2 + y^2)}{4ct} + \alpha\right)\right) \tag{18}$$

of equation (2).

Following in the same way, we get the similarity variables and forms for another element in the optimal system. The results are summarized in table 2.

Table 2. Similarity reductions of PDE (11) to complex ODEs.

Cases	p	$f(s,r)$	Reduced complex ODEs	
B_1	$\frac{s}{r}$	$\frac{1}{r}h(p)$	$(1+p)^2h''+4ph'+2h+\frac{a}{c}\lvert h\rvert^2h=0$	(2.1)
B_3+B_4	$r-s$	$h(p)$	$2ch''+a\lvert h\rvert^2h=0$	(2.2)
B_3+B_5	s	$h(p)\exp(\mathrm{i}r)$	$ch''-ch+a\lvert h\rvert^2h=0$	(2.3)
$B_3+B_4+B_5$	$s-r$	$h(p)\exp(\mathrm{i}r)$	$2ch''-2\mathrm{i}ch'-ch+a\lvert h\rvert^2h=0$	(2.4)

Equation (2.1) in table 2 for $\frac{c}{a}<0$ has a special solution

$$h(p)=\sqrt{\frac{-2c}{a}}\exp(\mathrm{i}\alpha)\qquad \alpha \text{ constant.} \tag{19}$$

Solution (19) leads by back substitution to the special solution

$$\psi(x,y,t)=\frac{1}{y}\sqrt{\frac{-2c}{a}}\exp\left(\mathrm{i}\left(\alpha+bt+\frac{(x^2+y^2)}{4ct}\right)\right) \tag{20}$$

of equation (2). Equation (2.2) in table 2 for $\frac{a}{c}<0$ has a special solution

$$h(p)=\frac{1}{\left(\gamma+p\sqrt{-\frac{a}{4c}}\right)}\exp(\mathrm{i}\alpha)\qquad \alpha,\gamma \text{ constants.} \tag{21}$$

From equations (21) and (11), we have

$$\psi(x,y,t)=\frac{1}{\left(\gamma t+(y-x)\sqrt{-\frac{a}{4c}}\right)}\exp\left(\mathrm{i}\left(\alpha+bt+\frac{(x^2+y^2)}{4ct}\right)\right). \tag{22}$$

Equation (2.3) in table 2 has a special solution

$$h(p)=\sqrt{\frac{2c}{a}}\left[\operatorname{sech}\left(\alpha\sqrt{\frac{2c}{a}}-p\right)\right]\exp(\mathrm{i}\gamma)\qquad a\neq 0,\alpha,\gamma \text{ constants.} \tag{23}$$

This implies that

$$\psi(x,y,t)=\frac{1}{t}\sqrt{\frac{2c}{a}}\left[\operatorname{sech}\left(\alpha\sqrt{\frac{2c}{a}}-\frac{x}{t}\right)\right]\exp\left(\mathrm{i}\left(\frac{y}{t}+\frac{(x^2+y^2)}{4ct}+\gamma+bt\right)\right) \tag{24}$$

is a solution of equation (2). Equation (2.4) in table 2 has the special solution

$$h(p)=\sqrt{\frac{c}{a}}\left[\operatorname{sech}\left(\frac{1}{2}(\gamma-p)\right)\right]\exp\left(\mathrm{i}\left(\frac{1}{2}p+\alpha\right)\right) \tag{25}$$

where γ,α are constants and $a\neq 0$. If we invert all our transformations, we have

$$\begin{aligned}\psi(x,y,t)=\frac{1}{t}\sqrt{\frac{c}{a}}&\left[\operatorname{sech}\left(\frac{1}{2}\left(\gamma-\frac{x}{t}+\frac{y}{t}\right)\right)\right]\\&\times\exp\left(\mathrm{i}\left(bt+\frac{1}{4ct}(x^2+y^2)+\frac{1}{2}\left(\frac{x}{t}+\frac{y}{t}\right)+\alpha\right)\right)\end{aligned} \tag{26}$$

as exact solution of equation (2).

Case 2. Corresponds to the PDE in type 2,

$$\mathrm{i}f_r+4cf_s+4csf_{ss}+a\lvert f\rvert^2f+bf=0. \tag{27}$$

Table 3. Similarity reductions of PDE (27) to complex ODE's.

Cases	p	$f(s,r)$	Reduced complex ODEs	
$B_2 - bB_3$	s	$h(p)\exp(\mathrm{i}br)$	$4cph'' + 4ch' + a\|h\|^2h = 0$	(3.1)
$B_1 - dB_3$	$\frac{s}{r^2}$	$\frac{1}{r}h(p)\exp\left(\mathrm{i}\left(br + \frac{1}{4cr}s - \frac{2}{r}d\right)\right)$	$2ph'' + 2h' - \frac{d}{c}h + \frac{a}{2c}\|h\|^2h = 0$	(3.2)

The essential reduction of this equation to ODEs can be derived by the optimal system $B_1 - dB_3$, $B_2 - bB_3$ where d is an arbitrary constant and

$$B_1 = rs\partial_s + \frac{1}{2}r^2\partial_r + \left(\mathrm{i}\left(\frac{s}{8c} + \frac{b}{2}r^2\right) - \frac{r}{2}\right)f\partial_f \qquad B_2 = \partial_r \qquad B_3 = -\mathrm{i}f\partial_f.$$

The corresponding reduced forms to ODEs are listed in table 3.

The first equation in table 3, for $\frac{c}{a} < 0$, has the special solution

$$h(p) = \sqrt{\frac{-c}{ap}}\exp(\mathrm{i}\alpha) \tag{28}$$

where α is constant.

Then, from equations (27) and (28), equation (2) has the special solution

$$\psi(x,y,t) = \sqrt{\frac{-c}{a(x^2+y^2)}}\exp(\mathrm{i}(\alpha + bt)). \tag{29}$$

The second equation in table 3 has the special solution

$$h(p) = \sqrt{\frac{2d}{a}}\exp(\mathrm{i}\alpha) \tag{30}$$

where α is constant.

Then from equations (27) and (30), we have

$$\psi(x,y,t) = \sqrt{\frac{2d}{a}}\frac{1}{t}\exp\left(\mathrm{i}\left(\alpha + bt + \frac{(x^2+y^2)}{4ct} - \frac{2d}{t}\right)\right) \tag{31}$$

as a special solution of equation (2).

Case 3. Corresponds to the PDE in type 3,

$$\mathrm{i}f_r + \frac{\mathrm{i}}{2r}f + cf_{ss} + a|f|^2f + bf = 0. \tag{32}$$

In the same way, the reduced ODEs which correspond to $B_1 + dB_2$ and $B_2 + dB_3$ are listed in table 4 where d is an arbitrary constant and

$$B_1 = r\partial_s + \frac{\mathrm{i}}{2c}sf\partial_f \qquad B_2 = \partial_s \qquad B_3 = -\mathrm{i}f\partial_f$$

The first equation in table 4 has a solution of the form

$$h(p) = \alpha p^{\frac{(2\mathrm{i}a\alpha^2 - d)}{2d}}(p+d)^{\frac{-(2\mathrm{i}a\alpha^2+d)}{2d}}\exp(\mathrm{i}(bp+\gamma)) \tag{33}$$

where α, γ are constants. Solution (33), together with equation (32), gives

$$\psi(x,y,t) = \alpha t^{\frac{(2\mathrm{i}a\alpha^2 - d)}{2d}}(t+d)^{\frac{-(2\mathrm{i}a\alpha^2+d)}{2d}}\exp\left(\mathrm{i}\left(bt + \gamma + \frac{y^2}{4ct} + \frac{x^2}{4c(t+d)}\right)\right). \tag{34}$$

The second equation in table 4 has the exact solution

$$h(p) = \alpha p^{\frac{(2\mathrm{i}a\alpha^2 - 1)}{2}}\exp(\mathrm{i}((b - cd^2)p + \gamma)) \tag{35}$$

Table 4. Similarity reductions of PDE (32) to complex ODE's.

Cases	p	$f(s,r)$	Reduced complex ODEs
$B_1 + dB_2$	R	$h(p)\exp\left(\frac{\mathrm{i}s^2}{4c(r+d)}\right)$	$\mathrm{i}h' + \left(\frac{\mathrm{i}}{2p} + b + \frac{\mathrm{i}}{2(p+d)}\right)h + a\lvert h\rvert^2 h = 0$ (4.1)
$B_2 + dB_3$	R	$h(p)\exp(-\mathrm{i}ds)$	$\mathrm{i}h' + \left(\frac{\mathrm{i}}{2p} + b - cd^2\right)h + a\lvert h\rvert^2 h = 0$ (4.2)

Table 5. Similarity reductions of PDE (37) to complex ODE's.

Cases	p	$f(s,r)$	Reduced complex ODEs	
$B_1 + 2(1-4cb)B_4$	s	$h(p)\exp\left(\frac{-\mathrm{i}}{4c}r(1-4cb)\right)$	$2ch'' + \mathrm{i}h' + a\lvert h\rvert^2 h = 0$	(5.1)
B_2	r	$h(p)\exp\left(\frac{\mathrm{i}}{8c}\left(\frac{s^2}{r} - 2s\right)\right)$	$\mathrm{i}h' + \left(\frac{\mathrm{i}}{2p} + b - \frac{1}{8c}\right)h + a\lvert h\rvert^2 h = 0$	(5.2)
B_3	r	$h(p)\exp\left(-\frac{\mathrm{i}}{4c}s\right)$	$\mathrm{i}h' + \left(b - \frac{1}{8c}\right)h + a\lvert h\rvert^2 h = 0$	(5.3)
$B_1 + B_3$	$r - s$	$h(p)\exp\left(-\frac{\mathrm{i}}{4c}r\right)$	$2ch'' + bh + a\lvert h\rvert^2 h = 0$	(5.4)
$B_1 + B_2$	$\frac{r^2}{2} - s$	$h(p)\exp\left(\mathrm{i}\left(-\frac{r^3}{12c} - \frac{r^2}{8c} + \frac{sr}{4c}\right)\right)$	$2ch'' - \mathrm{i}h' + \left(b - \frac{1}{4c}(1+p)\right)h + a\lvert h\rvert^2 h = 0$	(5.5)
$B_1 + B_3 - 2B_4$	$r - s$	$h(p)$	$2ch'' + \left(b - \frac{1}{4c}\right)h + a\lvert h\rvert^2 h = 0$	(5.6)

where α and γ are constants. The solution $h(p)$ leads by back substitution to the exact solutions of equation (2) of the form

$$\psi(x,y,t) = \alpha t^{\frac{(2\mathrm{i}a\alpha^2 - 1)}{2}} \exp\left(\mathrm{i}\left((b - cd^2)t - dx + \frac{y^2}{4ct} + \gamma\right)\right). \tag{36}$$

Case 4. Let us consider the PDE of type 4 in table 1,

$$4cf_{ss} + 2\mathrm{i}f_s + 2\mathrm{i}f_r + 2a\lvert f\rvert^2 f + \left(2b - \frac{1}{2c}\right)f = 0. \tag{37}$$

The reduced ODEs in this case will be obtained by the optimal system of six operators, listed in table 5, where the vector fields are

$$B_1 = \partial_r \qquad B_2 = r\partial_s + \frac{\mathrm{i}}{4c}(s-r)f\partial_f \qquad B_3 = \partial_s - \frac{\mathrm{i}}{4c}f\partial_f \qquad B_4 = -\frac{\mathrm{i}}{8c}f\partial_f.$$

The first equation in table 5 for $ac < 0$ has the solution of the form

$$h(p) = \frac{1}{\sqrt{-4ca}}\left[\sec\left(\frac{p}{4c}\right)\right]\exp\left(\mathrm{i}\left(\alpha - \frac{p}{4c}\right)\right) \tag{38}$$

where α is constant.

By using back transformation, we have the solution of equation (2) of the form

$$\psi(x,y,t) = \frac{1}{\sqrt{-4ca}}\left[\sec\left(\frac{(x-y)}{4c}\right)\right]\exp\left(\mathrm{i}\left(\alpha - \frac{(x+y)}{4c} - \frac{(1-4cb)t}{4c}\right)\right). \tag{39}$$

Equation (5.2) has exact solution

$$h(p) = \alpha p^{\mathrm{i}a\alpha^2 - \frac{1}{2}}\exp\left(\mathrm{i}\left(\gamma + \left(b - \frac{1}{8c}\right)p\right)\right) \tag{40}$$

where α, γ are constants. By using back transformation, we get

$$\psi(x,y,t) = \alpha t^{\mathrm{i}a\alpha^2 - \frac{1}{2}}\exp\left(\mathrm{i}\left(\gamma + \left(b - \frac{1}{8c}\right)t + \frac{1}{8c}\left(\frac{(x-y)^2}{t} - 2x - 2y\right)\right)\right) \tag{41}$$

as a solution of equation (2). Equation (5.3) has a special solution

$$h(p) = \alpha \exp\left(\mathrm{i}\left(\left(-\frac{1}{8c} + b + a\alpha^2\right) p + \gamma\right)\right) \tag{42}$$

where α, γ are constants. Then equation (2) has the solution

$$\psi(x, y, t) = \alpha \exp\left(\mathrm{i}\left(\left(-\frac{1}{8c} + b + a\alpha^2\right) t + \gamma - \frac{1}{4c}(x + y)\right)\right). \tag{43}$$

Equation (5.4) has the following solution for $b < 0$

$$h(p) = \sqrt{\frac{-2b}{a}}\left[\operatorname{sech}\left(\sqrt{\frac{-b}{2c}}(\gamma - p)\right)\right]\exp(\mathrm{i}\alpha) \tag{44}$$

where α, γ are constants. Then equation (2) has the solution

$$\psi(x, y, t) = \sqrt{\frac{-2b}{a}}\left[\operatorname{sech}\left(\sqrt{\frac{-b}{2c}}(\gamma + x - y - t)\right)\right]\exp\left(\mathrm{i}\left(\alpha - \frac{t}{4c} - \frac{y}{2c}\right)\right). \tag{45}$$

Equation (5.5) can be transformed to the second Painlevé equation

$$u'' = 2u^3 + zu \tag{46}$$

by using the transformation

$$\begin{cases} h(p) = \sqrt{\frac{-1}{a\sqrt[3]{c}}}\left[\exp\left(\mathrm{i}\left(\alpha + \frac{p}{4c}\right)\right)\right] u(z) \\ z = 2\sqrt[3]{c}\left(\frac{p}{4c} + \frac{1}{8c} - b\right) \end{cases} \qquad \text{for} \quad a\sqrt[3]{c} < 0 \tag{47}$$

The last equation in table 5 has the exact solution

$$h(p) = \sqrt{\frac{(1 - 4cb)}{2ca}}\left[\operatorname{sech}\left(-\sqrt{\frac{(1 - 4cb)}{8c^2}}(p + \alpha)\right)\right]\exp(\mathrm{i}\gamma) \tag{48}$$

where α and γ are constants. Then equation (2) has the solution

$$\psi(x, y, t) = \sqrt{\frac{(1 - 4cb)}{2ca}}\left[\operatorname{sech}\left(-\sqrt{\frac{(1 - 4cb)}{8c^2}}(y - x + t + \alpha)\right)\right]\exp\left(\mathrm{i}\left(\gamma - \frac{y}{2c}\right)\right). \tag{49}$$

Case 5. Corresponds to PDE of type 5 in table 1,

$$c(1 + d^2) f_{ss} + \mathrm{i} f_r + a|f|^2 f + bf = 0. \tag{50}$$

The reduced ODEs in this case will be obtained by the optimal system of six operators, listed in table 6, where the vector fields are

$$B_1 = \partial_r \qquad B_2 = r\partial_s + \frac{\mathrm{i}}{2c(1 + d^2)} sf\partial_f \qquad B_3 = \partial_s \qquad \text{and} \qquad B_4 = -\mathrm{i}f\partial_f$$

where $m = \frac{1}{\sqrt{c(1+d^2)}}$ and k, m are arbitrary constants.

Equation (6.1) in table 6 has the solution of the form

$$h(p) = \alpha p^{\mathrm{i}a\alpha^2 - \frac{1}{2}} \exp(\mathrm{i}(bp + \gamma)) \tag{51}$$

where α and γ are constants.

Table 6. Similarity reductions of PDE (50) to complex ODE's.

Cases	p	$f(s,r)$	Reduced complex ODEs	
B_2	r	$h(p)\exp\left(\frac{\mathrm{i}s^2}{4c(1+d^2)r}\right)$	$\mathrm{i}h' + \frac{\mathrm{i}}{2p}h + bh + a\|h\|^2h = 0$	(6.1)
$B_3 + mB_1$	$r - ms$	$h(p)$	$h'' + \mathrm{i}h' + bh + a\|h\|^2h = 0$	(6.2)
$4c^2B_2 + B_1$	$s - 2c^2r^2$	$h(p)\exp\left(\frac{2\mathrm{i}c}{(1+d^2)}\left(rs - \frac{2c^2r^3}{3}\right)\right)$	$(1+d^2)h'' - \frac{2}{(1+d^2)}ph + \frac{b}{c}h + \frac{a}{c}\|h\|^2h = 0$	(6.3)
$B_1 + kB_4$	s	$h(p)\exp(-\mathrm{i}kr)$	$(1+d^2)h'' + \frac{(k+b)}{c}h + \frac{a}{c}\|h\|^2h = 0$	(6.4)
$\sqrt{2c}B_3 + cB_4$	r	$h(p)\exp\left(-\mathrm{i}\sqrt{\frac{c}{2}}s\right)$	$2\mathrm{i}h' + (-c^2(1+d^2) + 2b)h + 2a\|h\|^2h = 0$	(6.5)
$\sqrt{2c}B_3 + B_1 + B_4$	$r - \frac{1}{\sqrt{2c}}s$	$h(p)\exp(-\mathrm{i}r)$	$(1+d^2)h'' + 2\mathrm{i}h' + 2(1+b)h + 2a\|h\|^2h = 0$	(6.6)

Then equation (2) has the solution

$$\psi(x,y,t) = \alpha t^{\mathrm{i}a\alpha^2 - \frac{1}{2}}\exp\left(\mathrm{i}\left(bt + \gamma + \frac{(x-dy)^2}{4c(1+d^2)t}\right)\right). \tag{52}$$

Equation (6.2) in table 6 has the exact solution

$$h(p) = \sqrt{\frac{-(4b+1)}{2a}}\left[\sec\left(\sqrt{\frac{(4b+1)}{4}}(p+\alpha)\right)\right]\exp\left(\mathrm{i}\left(\gamma - \frac{p}{2}\right)\right) \tag{53}$$

where $a < 0, b > 0$ and α, γ are constants. Solution (53) leads by back substitution to the exact solution of equation (2),

$$\psi(x,y,t) = \sqrt{\frac{-(4b+1)}{2a}}\left[\sec\left(\sqrt{\frac{(4b+1)}{4}}\left(t - \frac{(x-dy)}{\sqrt{c(1+d^2)}} + \alpha\right)\right)\right] \times \exp\left(\mathrm{i}\left(\gamma - \frac{t}{2} + \frac{(x-dy)}{2\sqrt{c(1+d^2)}}\right)\right). \tag{54}$$

Equation (6.3) can be transformed to the second Painlevé equation

$$u'' = 2u^3 + zu \tag{55}$$

by using the transformation

$$\begin{cases} h(p) = \sqrt{\frac{2c}{a}\sqrt[3]{\frac{-4}{(1+d^2)}}}[\exp(\mathrm{i}\alpha)]u(z) \\ z = \sqrt[3]{\frac{-(1+d^2)}{4}}\left(\frac{b}{c} - \frac{2p}{(1+d^2)}\right) \end{cases}. \tag{56}$$

Equation (6.4) in table 6 for $a < 0$ has the exact solution

$$h(p) = \sqrt{\frac{-2(b+k)}{a}}\left[\sec\left(\sqrt{\frac{(k+b)}{c(1+d^2)}}(p+\alpha)\right)\right]\exp(\mathrm{i}\gamma) \tag{57}$$

where γ and α are constants. Then equation (2) has the solution

$$\psi(x,y,t) = \sqrt{\frac{-2(b+k)}{a}}\left[\sec\left(\sqrt{\frac{(k+b)}{c(1+d^2)}}(x - dy + \alpha)\right)\right]\exp(\mathrm{i}(\gamma - kt)). \tag{58}$$

Equation (6.5) in table 6 has a special solution of the form

$$h(p) = \alpha\exp\left(\mathrm{i}\left(\left(a\alpha^2 + b - \frac{c^2(1+d^2)}{2}\right)p + \gamma\right)\right) \tag{59}$$

Table 7. Similarity reductions of PDE (63) to complex ODE's.

Cases	p	$f(s,r)$	Reduced complex ODEs	
B_1	(s^2+r^2)	$h(p)$	$4ph''+4h'+\frac{k}{c^2}h+\frac{a}{c}\lvert h\rvert^2h=0$	(7.1)
$B_3-B_4+dB_2$	$s-dr$	$h(p)\exp(\mathrm{i}r)$	$(1+d^2)h''-2\mathrm{i}dh'+\left(\frac{k}{c^2}-1\right)h+\frac{a}{c}\lvert h\rvert^2h=0$	(7.2)
B_3+dB_2	$s-dr$	$h(p)$	$(1+d^2)h''+\frac{k}{c^2}h+\frac{a}{c}\lvert h\rvert^2h=0$	(7.3)

where α and γ are constants. Then equation (2) has the solution

$$\psi(x,y,t)=\alpha\exp\left(\mathrm{i}\left(\left(a\alpha^2+b-\frac{c^2(1+d^2)}{2}\right)t+\gamma-\sqrt{\frac{c}{2}}(x-dy)\right)\right). \tag{60}$$

Equation (6.6) in table 6 for $a<0$ and $b>0$ has the exact solution

$$h(p)=\sqrt{\frac{-(1+d^2)\beta}{a}}[\sec(-\sqrt{\beta}(p+\gamma))]\exp\left(\mathrm{i}\left(\alpha-\frac{p}{(1+d^2)}\right)\right) \tag{61}$$

where $\sqrt{\beta}=\sqrt{\frac{(1+2(1+b)(1+d^2))}{(1+d^2)^2}}$. Then equation (2) has the solution

$$\psi(x,y,t)=\sqrt{\frac{-(1+d^2)\beta}{a}}\left[\sec\left(-\sqrt{\beta}\left(\gamma+t-\frac{(x-dy)}{\sqrt{2c}}\right)\right)\right]$$
$$\times\exp\left(\mathrm{i}\left(\alpha-t-\frac{(t\sqrt{2c}-x+dy)}{\sqrt{2c}(1+d^2)}\right)\right). \tag{62}$$

Case 6. The PDE of type 6 in table 1,

$$cf_{rr}+cf_{ss}+\frac{k}{c}f+a|f|^2f=0. \tag{63}$$

The complex ordinary differential equations which correspond to the basic fields of an optimal system given by $B_1, dB_2+B_3-B_4, dB_2+B_3$ are listed in table 7, where d is an arbitrary constant and

$$B_1=r\partial_s-s\partial_r \qquad B_2=\partial_s \qquad B_3=\partial_r \qquad B_4=\mathrm{i}f\partial_f.$$

The first equation in table 7, for $\frac{k}{ca}<0$, has the exact solution

$$h(p)=\sqrt{\frac{-k}{ca}}\exp(\mathrm{i}\gamma) \tag{64}$$

where γ is an arbitrary constant. Then equation (2) has the solution

$$\psi(x,y,t)=\sqrt{\frac{-k}{ca}}\frac{1}{t}\exp\left(\mathrm{i}\left(\gamma+bt+\frac{k}{ct}+\frac{(x^2+y^2)}{4ct}\right)\right). \tag{65}$$

The second equation in table 7 has the special solution

$$h(p)=\sqrt{\frac{2c^2-2k(1+d^2)}{ac(1+d^2)}}\left[\operatorname{sech}\left(-\sqrt{\frac{(c^2-k(1+d^2))}{c^2(1+d^2)^2}}(p+\gamma)\right)\right]$$
$$\times\exp\left(\mathrm{i}\left(\alpha+\frac{dp}{(1+d^2)}\right)\right) \tag{66}$$

Table 8. Similarity reductions of PDE (70) to complex ODE's.

Cases	p	$f(s,r)$	Reduced complex ODEs
B_1	$\sqrt{s^2+r^2}$	$h(p)\exp\left(\frac{\mathrm{i}}{2c}(\sigma s+\lambda r)\right)$	$h''+\frac{1}{p}h'+\frac{(\sigma^2+\lambda^2)}{4c^2}h+\frac{a}{c}\lvert h\rvert^2h=0$ (8.1)
B_2	r	$h(p)\exp\left(\frac{\mathrm{i}\sigma s}{2c}\right)$	$h''-\frac{\mathrm{i}\lambda}{c}h'+\frac{\sigma^2}{4c^2}h+\frac{a}{c}\lvert h\rvert^2h=0$ (8.2)
$\frac{1}{2c}(\lambda B_2-\sigma B_3)$	$-\frac{1}{2c}(\sigma s+\lambda r)$	$h(p)$	$(\sigma^2+\lambda^2)h''+2\mathrm{i}(\sigma^2+\lambda^2)h'+4ca\lvert h\rvert^2h=0$ (8.3)
B_2+B_3	$s-r$	$h(p)\exp\left(\frac{\mathrm{i}}{2c}(\sigma+\lambda)s\right)$	$8c^2h''+8\mathrm{i}c\lambda h'+(\sigma^2-\lambda^2)h+4ca\lvert h\rvert^2h=0$ (8.4)
$B_1+\frac{1}{2c}(\lambda B_2-\sigma B_3)$	$\sqrt{\left(s+\frac{\sigma}{2c}\right)^2+\left(r+\frac{\lambda}{2c}\right)^2}$	$h(p)\exp\left(\frac{\mathrm{i}}{4c^2}(2c(\sigma s+\lambda r)+\lambda^2)\right)$	$4c^2h''+\frac{4c^2}{p}h'+(\lambda^2+\sigma^2)h+4ca\lvert h\rvert^2h=0$ (8.5)

where α, γ are arbitrary constants. Then equation (2) has the solution

$$\psi(x,y,t)=\frac{1}{t}\sqrt{\frac{2c^2-2k(1+d^2)}{ac(1+d^2)}}\left[\operatorname{sech}\left(-\sqrt{\frac{(c^2-k(1+d^2))}{c^2(1+d^2)^2}}\left(\gamma+\frac{x}{t}-\frac{dy}{t}\right)\right)\right]$$
$$\times\exp\left(\mathrm{i}\left(\alpha+\frac{d}{(1+d^2)}\left(\frac{x}{t}-\frac{dy}{t}\right)+bt+\frac{y}{t}+\frac{k}{ct}+\frac{(x^2+y^2)}{4ct}\right)\right). \tag{67}$$

The third equation in table 7 has the exact solution of the form

$$h(p)=2\left[\operatorname{sech}\left(\sqrt{\frac{2a}{c(1+d^2)}}(\gamma-p)\right)\right]\exp(\mathrm{i}\alpha) \tag{68}$$

where $k=-2ca$ and α, γ are arbitrary constants. Then equation (2) has the solution

$$\psi(x,y,t)=\frac{2}{t}\left[\operatorname{sech}\left(\sqrt{\frac{2a}{c(1+d^2)}}\left(\gamma-\frac{x}{t}+\frac{dy}{t}\right)\right)\right]$$
$$\times\exp\left(\mathrm{i}\left(\alpha+bt+\frac{(x^2+y^2-8ca)}{4ct}\right)\right). \tag{69}$$

Case 7. Corresponds to the PDE of type 7 in table 1,

$$cf_{ss}-\mathrm{i}\sigma f_s-\mathrm{i}\lambda f_r+cf_{rr}+a|f|^2f=0. \tag{70}$$

The vector fields in this case are

$$B_1=r\partial_s-s\partial_r+\frac{\mathrm{i}}{2c}(\sigma r-\lambda s)f\partial_f \qquad B_2=\partial_s+\frac{\mathrm{i}}{2c}\sigma f\partial_f \qquad B_3=\partial_r+\frac{\mathrm{i}}{2c}\lambda f\partial_f.$$

The essential reduction is given in table 8.

Equation (8.1) in table 8 has a special solution for $ac<0$ of the form

$$h(p)=\sqrt{\frac{-(\lambda^2+\sigma^2)}{4ca}}\exp(\mathrm{i}\alpha) \tag{71}$$

where α is constant. Then equation (2) has the solution

$$\psi(x, y, t) = \sqrt{\frac{-(\lambda^2+\sigma^2)}{4ca}} \exp\left(\frac{\mathrm{i}}{2c}((2cb - \lambda^2 - \sigma^2)t + \sigma x + \lambda y + 2c\alpha)\right). \tag{72}$$

Equation (8.2) has the exact solution for $ca < 0$ of the form

$$h(p) = \sqrt{\frac{(\lambda^2+\sigma^2)}{-2ca}} \left[\sec\left(\sqrt{\frac{(\lambda^2+\sigma^2)}{4c^2}}(p+\gamma)\right)\right] \exp\left(\mathrm{i}\left(\alpha + \frac{\lambda p}{2c}\right)\right) \tag{73}$$

where α, γ are arbitrary constants. Then equation (2) has the solution

$$\psi(x, y, t) = \sqrt{\frac{-(\lambda^2+\sigma^2)}{2ca}} \left[\sec\left(\sqrt{\frac{\lambda^2+\sigma^2}{4c^2}}(y - \lambda t + \gamma)\right)\right] \\ \times \exp\left[\frac{\mathrm{i}}{2c}\{2ca + \lambda(y - \lambda t) + \sigma(x - \sigma t) + 2cbt\}\right] \tag{74}$$

Equation (8.3) in table 8 for $ca < 0$ has the solution of the form

$$h(p) = \sqrt{\frac{(\lambda^2+\sigma^2)}{2ca}}[\sec(p+\gamma)] \exp(\mathrm{i}(-p+\alpha)) \tag{75}$$

where α, γ are arbitrary constants. Then equation (2) has the solution

$$\psi(x, y, t) = \sqrt{-\frac{(\lambda^2+\sigma^2)}{2ca}} \left[\sec\left(\frac{-\sigma}{2c}(x - \sigma t) - \frac{\lambda}{2c}(y - \lambda t) + \gamma\right)\right] \\ \times \exp\left(\mathrm{i}\left(\alpha + \frac{\sigma}{2c}(x - \sigma t) + \frac{\lambda}{2c}(y - \lambda t) + bt\right)\right). \tag{76}$$

Equation (8.4) in table 8 for $ca < 0$ has the solution of the form

$$h(p) = \sqrt{\frac{-(\lambda^2+\sigma^2)}{2ca}} \left[\sec\left(\sqrt{\frac{(\lambda^2+\sigma^2)}{8c^2}}(p+\gamma)\right)\right] \exp\left(\mathrm{i}\left(\alpha - \frac{\lambda p}{2c}\right)\right) \tag{77}$$

where α, γ are arbitrary constants. Then equation (2) has the solution

$$\psi(x, y, t) = \sqrt{\frac{-(\lambda^2+\sigma^2)}{2ca}} \left[\sec\left(\sqrt{\frac{(\lambda^2+\sigma^2)}{8c^2}}(x - y + (\lambda - \sigma)t + \gamma)\right)\right] \\ \times \exp\left(\frac{\mathrm{i}}{2c}(2c\alpha + \sigma x + \lambda y + (2bc - \lambda^2 - \sigma^2)t)\right). \tag{78}$$

Equation (8.5) in table 8 for $ca < 0$ has the solution of the form

$$h(p) = \sqrt{\frac{-(\lambda^2+\sigma^2)}{4ca}} \exp(\mathrm{i}\alpha) \tag{79}$$

where α is constant. Then equation (2) has the solution

$$\psi(x, y, t) = \sqrt{\frac{-(\lambda^2+\sigma^2)}{4ca}} \exp\left(\frac{\mathrm{i}}{4c^2}[(4c^2b - 2c(\lambda^2+\sigma^2))t + 2c(\sigma x + \lambda y) + \lambda^2 + 4c^2\alpha]\right). \tag{80}$$

Case 8. For the PDE of type 8 in table 1,

$$c\left(1 + \frac{1}{e^2r^2}\right) f_{ss} + \mathrm{i} f_r + \frac{\mathrm{i}}{2r} f + a|f|^2 f + bf = 0. \tag{81}$$

Table 9. Similarity reductions of PDE (89) to complex ODE's.

Cases	p	$f(s,r)$	Reduced complex ODEs	
B_3	(s^2+r^2)	$h(p)$	$8cph''+(-2\mathrm{i}p+8c)h'-\mathrm{i}h+2a\lvert h\rvert^2h=0$	(9.1)
B_4	r	$h(p)\exp\left(\frac{\mathrm{i}}{4c}s^2\right)$	$2ch''-\mathrm{i}ph'+2a\lvert h\rvert^2h=0$	(9.2)
B_1-dB_2	$ds+r$	$h(p)$	$2c(1+d^2)h''-\mathrm{i}ph'-\mathrm{i}h+2a\lvert h\rvert^2h=0$	(9.3)

We have two vectors

$$B_1=\partial_s \qquad \text{and} \qquad B_2=\mathrm{i}f\partial_f$$

the linear combination B_1+B_2 leads to the ansatz

$$p=r \qquad f(r,s)=h(p)\exp(\mathrm{i}s)$$

where $h(p)$ satisfies the following equation:

$$\mathrm{i}h'+\left(b-c-\frac{c}{e^2p^2}+\frac{\mathrm{i}}{2p}\right)h+a|h|^2h=0 \tag{82}$$

which has the solution of the form

$$h(p)=\frac{a}{p^{-\mathrm{i}a\alpha^2+\frac{1}{2}}}\exp\left(\mathrm{i}\left(\gamma+(b-c)p+\frac{c}{e^2p}\right)\right) \tag{83}$$

where α, γ are constants. Equation (83) leads by back substitution to the exact solution of equation (2),

$$\psi(x,y,t)=\frac{\alpha}{t^{-\mathrm{i}a\alpha^2+\frac{1}{2}}}\exp\left(\mathrm{i}\left(\gamma+(b-c)t+\frac{c}{e^2t}+x-\frac{y}{et}+\frac{y^2}{4ct}\right)\right). \tag{84}$$

Case 9. For the PDE of type 9 in table 1,

$$2cf_{rr}+2cf_{ss}-2\mathrm{i}f_s-\frac{1}{C}rf+2a|f|^2f=0. \tag{85}$$

Applying the symmetry method, we have two vector fields $B_1=\partial_s$, $B_2=\mathrm{i}f\partial_f$, where the linear combination $B_1+\frac{1}{c}B_2$ leads to the ansatz

$$p=r \qquad \text{and} \qquad f(r,s)=h(p)\exp\left(\frac{\mathrm{i}s}{c}\right)$$

where $h(p)$ satisfies the following equation:

$$h''-\frac{p}{2c^2}h+\frac{a}{c}|h|^2h=0. \tag{86}$$

In equation (86) assuming that $h(p)$ is real and using the transformation

$$\rho=\sqrt[3]{4c}\frac{p}{2c} \qquad \text{and} \qquad h(p)=\sqrt{\frac{-2}{a\sqrt[3]{4c}}}u(\rho) \qquad \text{for} \quad a\sqrt[3]{c}<0 \tag{87}$$

we have

$$u''=\rho u+2u^3 \tag{88}$$

which is the second Painlevé equation.

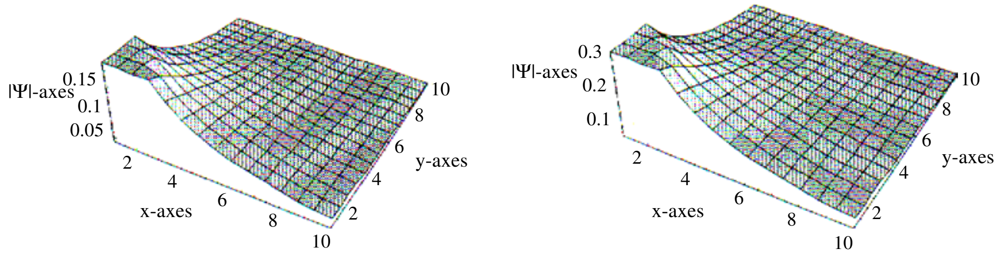
Case 10. For the PDE of type 10 in table 1,

$$2cf_{rr}+2cf_{ss}-\mathrm{i}rf_r-\mathrm{i}f-\mathrm{i}sf_s+2a|f|^2f=0. \tag{89}$$

The vector fields in this case are

$$B_1=\partial_s \qquad B_2=\partial_r \qquad B_3=r\partial_s-s\partial_r \qquad \text{and} \qquad B_4=\partial_s+\frac{\mathrm{i}}{2c}sf\partial_f$$

the essential reductions are given in table 9.



(*a*) (*b*)

Figure 1. Amplitude $|\psi(x, y, t)|$ of the exact solution (18) of (*a*) equation (2) with $c = -1$, $a = 1$, $b = 2\mathrm{i}$ and $t = 0.25$ and (*b*) equation (1) with $c = -1$, $a = 1$, $b = 0$ and $t = 0.25$.

The first ODE in table 9, for $\frac{c}{a} < 0$, has the solution

$$h(p) = \sqrt{\frac{-c}{ap}}\exp(\mathrm{i}\alpha) \tag{90}$$

where α is an arbitrary constant. Then equation (89) has the exact solution

$$\psi(x, y, t) = \sqrt{\frac{-c}{a((y-2t)^2 + (x-2t)^2)}}\exp\left(\mathrm{i}\left(\alpha + bt + \frac{1}{c}(x + y - 2t)\right)\right). \tag{91}$$

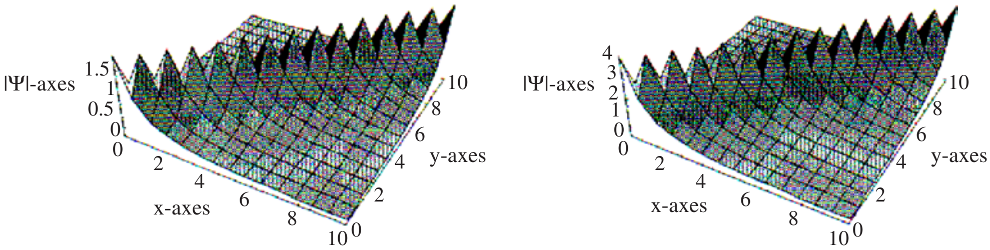
We cannot find an analytic solution for equations (9.2) and (9.3).

4. Discussion and concluding remarks

We have attempted to find comprehensive analytical solutions to cubic NLS equation (2) by applying the Lie group method. Let us comment on some of the qualitative features of these analytic solutions: the explicit forms of the complex function $\psi(x, y, t)$ contain the physical constants (a, b, c), which may enable one to discuss the behaviour of $\psi(x, y, t)$ as a function of these constants and this also provides enough freedom to build up solutions that may correspond to a particular physical situation. Also, these exact solutions of equation (2) contain some arbitrary constants of integration $(\alpha, \gamma, d, \lambda, \sigma, k)$, which can be chosen so that $\psi(x, y, t)$ simulates some desired physical situation, or initial conditions $\psi(x, y, 0)$ have some desired features, which means a great variation in the solutions. In order to avoid non-physical solutions, one has to choose the arbitrary constants in a suitable manner. This feature is characteristic of all time-dependent solutions of NLS equation (2) we obtained. On the other hand, it is of interest to note that the explicit dependence of these solutions on the constant b will illustrate the connection between NLS equations (1) and (2). It is important to emphasize that the exact solution of NLS equation (1) can be derived from the corresponding solutions of NLS equation (2) by letting $b = 0$.

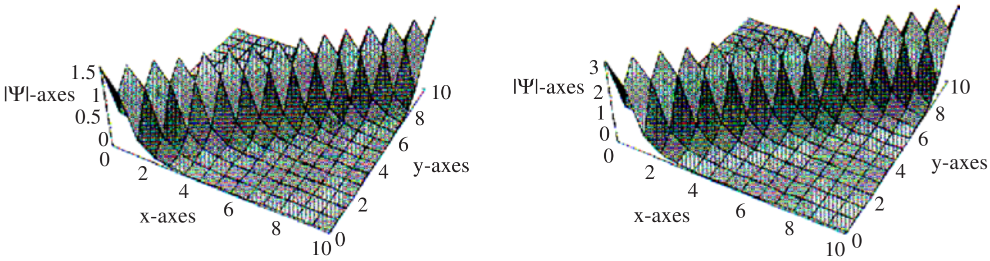
In a complementary approach, to sketch the features of the typical behaviour of some exact analytic solutions of NLS equation (2) and the corresponding solutions of NLS equation (1) (where $b = 0$), some of these solutions are plotted in figures 1–10 with random choices both of values of the arbitrary constants $(\alpha, \gamma, d, \lambda, \sigma, k)$ and of the physical constants (a, b, c).

The plots illustrate connections between the absolute value of the function $\psi(x, y, t)$ (amplitude $|\psi(x, y, t)|$) and the constant value b. The structure of the solution amplitude plotted in figures 1 and 5 exhibits a dissipating behaviour. It sometimes looks like a hump, the



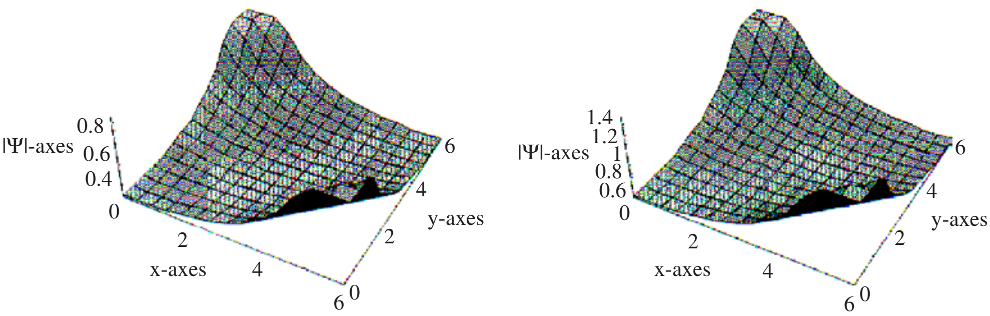
(*a*) (*b*)

Figure 2. Amplitude $|\psi(x, y, t)|$ of the exact solution (22) of (*a*) equation (2) with $c = -1$, $a = \gamma = 1$, $b = 3\text{i}$ and $t = 0.25$ and (*b*) equation (1) with $c = -1$, $a = \gamma = 1$, $b = 0$ and $t = 0.25$.



(*a*) (*b*)

Figure 3. Amplitude $|\psi(x, y, t)|$ of the exact solution (26) of (*a*) equation (2) with $c = a = \gamma = 1$, $b = 3\text{i}$ and $t = 0.25$ and (*b*) equation (1) with $c = a = \gamma = 1$, $b = 0$ and $t = 0.25$.



(*a*) (*b*)

Figure 4. Amplitude $|\psi(x, y, t)|$ of the exact solution (39) of (*a*) equation (2) with $c = -1$, $a = \gamma = 1$, $b = 2\text{i}$ and $t = 0.25$ and (*b*) equation (1) with $c = -1$, $a = \gamma = 1$, $b = 0$ and $t = 0.25$.

amplitude of such a structure is shown in figure 6, and the others exhibit a periodic peaking with saw-toothed appearance. The high of these peaks can be managed by a suitable choice of the arbitrary constants that appear in the formal solutions.

Although the plotted amplitudes $|\psi(x, y, t)|$ do not have the same profile for different solutions, their common feature is that the constant b plays the role of an inhibitory agent, sometimes it decreases (or diminishes) $|\psi(x, y, t)|$ by half. When b is very small, the plotted amplitude $|\psi(x, y, t)|$ does not show any significant difference between NLS equations (1) and (2), for any of our exact analytic solutions.

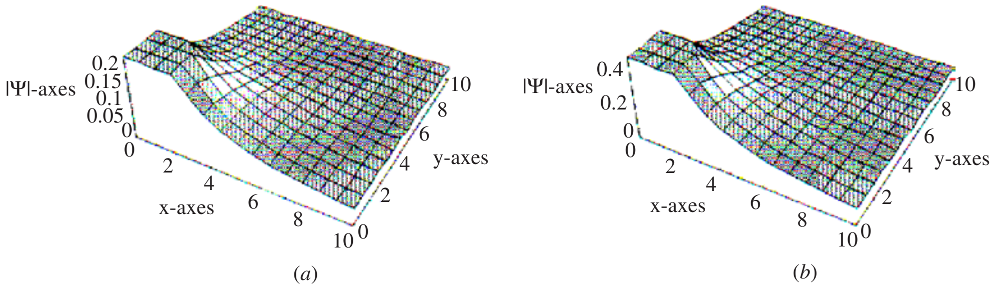


Figure 5. Amplitude $|\psi(x, y, t)|$ of the exact solution (41) of (*a*) equation (2) with $c = a = \gamma = 1$, $b = 3\mathrm{i}$ and $t = 0.25$ and (*b*) equation (1) with $c = a = \gamma = 1$, $b = 0$ and $t = 0.25$.

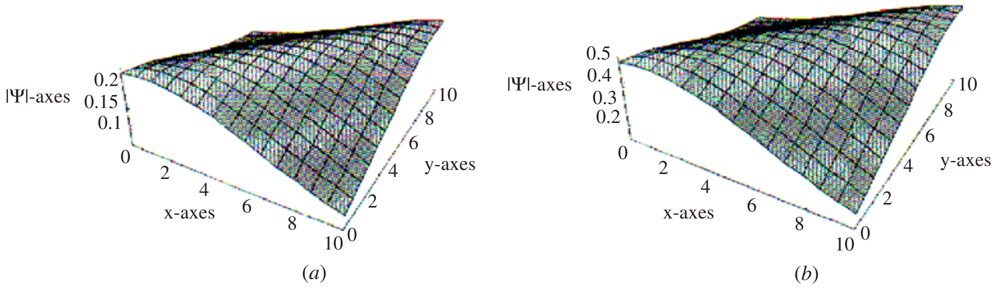


Figure 6. Amplitude $|\psi(x, y, t)|$ of the exact solution (49) of (*a*) equation (2) with $a = \alpha = 1$, $c = 2$, $b = 3\mathrm{i}$ and $t = 0.25$ and (*b*) equation (1) with $a = \alpha = 1$, $c = 2$, $b = 0$ and $t = 0.25$.

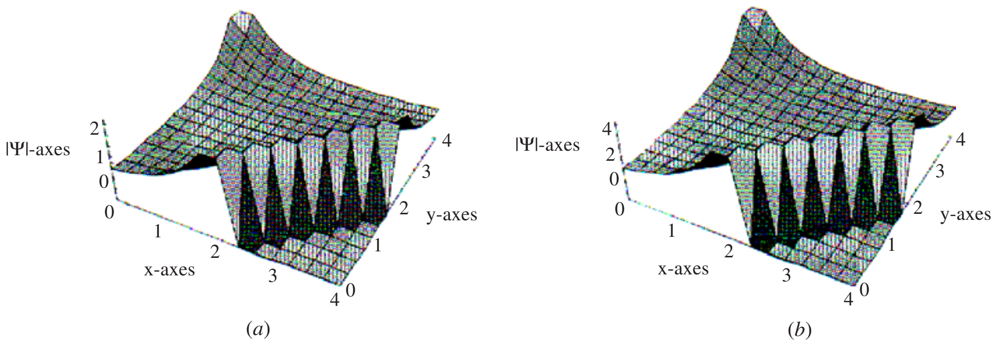
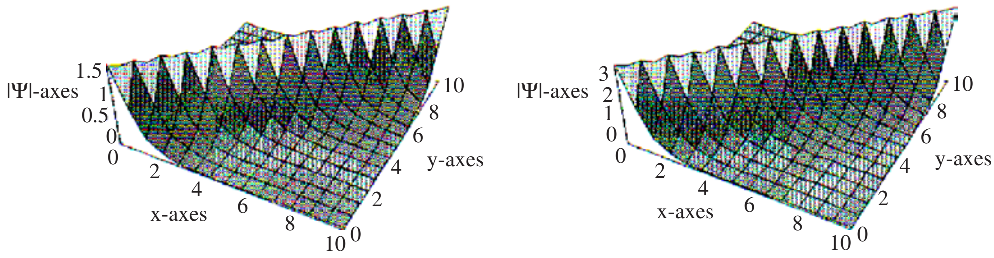


Figure 7. Amplitude $|\psi(x, y, t)|$ of the exact solution (58) of (*a*) equation (2) with $a = -1$, $\alpha = k = d = 1$, $c = 2$, $b = 3\mathrm{i}$ and $t = 0.25$ and (*b*) equation (1) with $a = -1$, $\alpha = k = d = 1$, $c = 2$, $b = 0$ and $t = 0.25$.

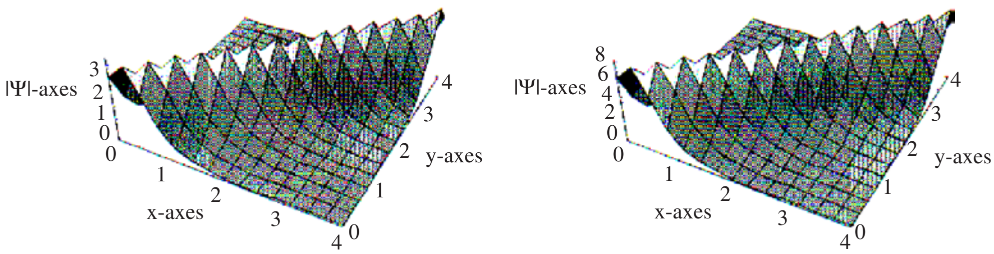
As a final comment, although it is not our present intention to discuss some definite physical phenomena, one may note that the above qualitative picture of the analytic solutions of NLS equation (2) is consistent with the rigorous analysis [26, 41–43] of the NLS model (2) for beam propagation through a Kerr medium, when linear damping (last term in equation (2)) is included where they conclude that the linear damping acts as a defocusing mechanism which delays the onset of blowup and may even arrest it.

However, linear damping arrests blowup only when it is not sufficiently small, otherwise it does not prevent the singularity formation. This kind of effect on singularity formation



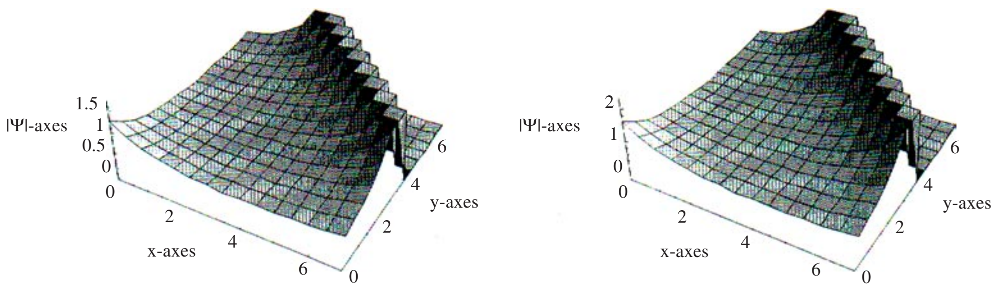
(*a*) (*b*)

Figure 8. Amplitude $|\psi(x, y, t)|$ of the exact solution (67) of (*a*) equation (2) with $c = a = \gamma = d = k = 1$, $b = 3\mathrm{i}$ and $t = 0.25$ and (*b*) equation (1) with $c = a = \gamma = d = k = 1$, $b = 0$ and $t = 0.25$.



(*a*) (*b*)

Figure 9. Amplitude $|\psi(x, y, t)|$ of the exact solution (69) of (*a*) equation (2) with $a = \gamma = d = 1$, $c = 2$, $b = 3\mathrm{i}$ and $t = 0.25$ and (*b*) equation (1) with $a = \gamma = d = 1$, $c = 2$, $b = 0$ and $t = 0.25$.



(*a*) (*b*)

Figure 10. Amplitude $|\psi(x, y, t)|$ of the exact solution (76) of (*a*) equation (2) with $a = -1$, $\gamma = \lambda = \sigma = 1$, $c = 2$, $b = \mathrm{i}$ and $t = 0.25$ and (*b*) equation (1) with $a = -1$, $\gamma = \lambda = \sigma = 1$, $c = 2$, $b = 0$ and $t = 0.25$.

distinguishes linear damping from all other defocusing perturbations of NLS analyzed so far using modulation theory [11].

Acknowledgments

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